

Analytic approach to the thermal Casimir force between metal and dielectric

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Abstract

The analytic asymptotic expressions for the Casimir free energy, pressure and entropy at low temperature in the configuration of one metal and one dielectric plate are obtained. For this purpose we develop the perturbation theory in a small parameter proportional to the product of the separation between the plates and the temperature. This is done using both the simplified model of an ideal metal and of a dielectric with constant dielectric permittivity and for the realistic case of the metal and dielectric with frequency-dependent dielectric permittivities. The analytic expressions for all related physical quantities at high temperature are also provided. The obtained analytic results are compared with numerical computations and good agreement is found. We demonstrate for the first time that the Lifshitz theory, when applied to the configuration of metal-dielectric, satisfies the requirements of thermodynamics if the static dielectric permittivity of a dielectric plate is finite. If it is infinitely large, the Lifshitz formula is shown to violate the Nernst heat theorem. The implications of these results for the thermal quantum field theory in Matsubara formulation and for the recent measurements of the Casimir force between metal and semiconductor surfaces are discussed.

Key words: Casimir force; Thermal corrections; Lifshitz formula; Nernst heat theorem;

PACS: 03.70.+k, 11.10.Wx, 12.20.-m, 12.20.Ds

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1. Introduction

The Casimir effect [1] is the direct manifestation of zero-point oscillations of quantized fields. It finds multidisciplinary applications in quantum field theory, gravitation and cosmology, atomic physics, condensed matter and, most recently, in nanotechnology (see, e.g., the monographs [2,3,4,5] and reviews [6,7,8,9]). According to Casimir's prediction, the existence of zero-point oscillations leads to the polarization of vacuum in quantization volumes restricted by material boundaries and in spaces with non-Euclidean topology. This is accompanied by forces acting on the boundary surfaces (the so called Casimir force). The Casimir force acts between electrically neutral closely spaced surfaces. It is a pure quantum phenomenon (there is no such a force in the framework of classical electrodynamics) being the generalization of the well known van der Waals force for the case of relatively large separations where relativistic effects become essential.

The theoretical basis for the description of both the van der Waals and Casimir forces is given by the Lifshitz theory [10,11,12]. The main formulas of the Lifshitz theory express the free energy and pressure of the van der Waals and Casimir interaction between two plane parallel plates as some functionals of the frequency-dependent dielectric permittivities of plate materials. These formulas can be derived in many different theoretical schemes [2,3,4,5,6,7,8,9,10,11,12]. In particular, they were obtained in the framework of thermal quantum field theory in the Matsubara formulation [8]. During the last few years the Lifshitz theory was successfully applied to the interpretation of many measurements of the Casimir force between metal surfaces [13,14,15,16,17,18,19,20,21,22,23,24] and between metal and semiconductor [25,26,27,28].

A complicated problem of the Lifshitz theory is how to describe the Casimir interaction between real metals at nonzero temperature. The most convenient form of the Lifshitz formulas exploits the dielectric permittivity along the imaginary frequency axis. The latter is obtained from the tabulated optical data for the complex index of refraction by means of the Kramers-Kronig relations. The available data are, however, insufficient and must be extrapolated in some way to lower frequencies. In this respect the contribution from the zero frequency is of most concern. The point is that in Matsubara thermal field theory the zero-frequency term becomes dominant at large separations (high temperatures) whereas the contributions from all other Matsubara frequencies being exponentially small. In [29,30,31] the zero-frequency term of the Lifshitz formula was obtained by using the dielectric function of the Drude model. This results in a violation of the Nernst heat theorem in the case of perfect crystal lattices [32,33] and is in contradiction with experiments at separations below $1\mu\text{m}$ [22,23,24,34]. The asymptotic value of the Casimir force at large sepa-

rations predicted in [29,30,31] is equal to one half of the value predicted for ideal metals, i.e., to one half of the so called classical limit [35,36].

Another approach [37,38] uses the dielectric permittivity of the plasma model to determine the zero-frequency term of the Lifshitz formula. This approach was shown to be in agreement with thermodynamics [32,33] and consistent with experiment [23]. It predicts the magnitudes of the Casimir force at short separations in qualitative agreement with the case of ideal metals. At large separations the predicted force magnitude is practically equal to that for ideal metals. Very similar results, which are also in agreement with the requirements of thermodynamics and consistent with experiment, are predicted by the surface impedance approach [39,40]. The controversies among different approaches to the thermal Casimir force between metals are detailedly discussed in [31,33,40,41,42,43,44].

Recently it was demonstrated [45,46,47] that even the traditional application of the Lifshitz formula to the case of two dielectric semispaces presents problems. In [45,46,47] the analytic asymptotic expressions for the Casimir free energy, pressure and entropy at low temperatures (short separations) were found for two dielectrics. It was shown that if the dielectric materials possess finite static dielectric permittivities the theory is self-consistent and in agreement with thermodynamics. If, however, a nonzero dc conductivity of dielectrics is taken into account (any dielectric at nonzero temperature is characterized by some nonzero dc conductivity which is many orders of magnitude lower than for metals), this leads to a qualitative enhancement of the Casimir force and a simultaneous violation of the Nernst heat theorem. (Note that the dc conductivity of dielectrics was taken into account in [48] to explain the large observed effect in noncontact friction [49].) In [45,46,47] the phenomenological prescription was proposed that the dc conductivity of dielectrics is not related to the Casimir interaction, and to avoid contradictions with thermodynamics it should not be included in the model of dielectric response.

The difficulties which were met in the application of the Lifshitz theory to two metal and two dielectric plates attracted attention to the case when one plate is metallic and another one dielectric. This configuration was first considered in [50]. It presents the interesting opportunity to investigate the Casimir force in the case when different plates are described by quite different models of the dielectric response. In [50], however, only the first leading terms in the low-temperature asymptotic expressions for the Casimir free energy and entropy were obtained, and the pressure was derived only in the dilute approximation. In the analytical derivations in [50] (see also the review [47]) it was supposed that the metallic plate is made of ideal metal and the dielectric of the other plate is described by the frequency independent dielectric permittivity. These suppositions narrow the applicability of the obtained results. Also, the role of the dc conductivity of a dielectric plate was not investigated for plates with

frequency-dependent dielectric permittivities.

In the present paper we develop the analytic approach to the thermal Casimir force acting between metal and dielectric permitting to find several expansion terms in the asymptotic expressions for all physical quantities at low temperature. This approach is applied not only to the configuration of ideal metal and dielectric with frequency independent dielectric permittivity but also to real metal and dielectric described by the dielectric permittivities depending on frequency. We pioneer in derivation of the low-temperature asymptotic expressions for the Casimir free energy, entropy and pressure between real metal and dielectric. The asymptotic behavior of all physical quantities at high temperatures (large separations) is also provided. What is more, the obtained representation for the Casimir free energy permits to find the low-temperature behavior of the Casimir force acting between a metal sphere and a dielectric plate (or, alternatively, dielectric sphere above a metal plate). This can be done with the help of the proximity force theorem [51]. The configuration of a sphere above a plate is most topical in experiments on the measurement of the Casimir force [13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28]. (Note that for the experimental parameters the error introduced by the use of the proximity force theorem was recently shown to be less than 0.1% [52,53,54,55].) Thus, our results will find immediate utility in experiment. The analytic expressions for the Casimir interaction between metal and dielectric at zero temperature are also found here for the first time. We determine the applicability region of the obtained analytic formulas by comparing them with numerical computations using the tabulated optical data for metallic and dielectric materials. The fundamental conclusion following from our results is that the Lifshitz theory, applied to the configuration of a metal and a dielectric plate, is in agreement with the Nernst heat theorem if the static dielectric permittivity of a dielectric plate is finite. Note that this conclusion could not be achieved by using the numerical computations which inevitably identify zero with all nonzero numbers in the limits of a computational error. If, however, the dc conductivity of a dielectric plate is included in the model of dielectric response, we show that the Nernst heat theorem is violated. This is in analogy to the same conclusion in [45] obtained for the configuration of two dielectric plates and confirms our phenomenological prescription that the dc conductivity is not related to the van der Waals and Casimir forces and should not be included in the model of dielectric response. Recently this prescription was confirmed experimentally [28].

The paper is organized as follows. In Section 2 we summarize the main formulas of the Lifshitz theory for the configuration of one plate made of metal and another one made of dielectric. Section 3 is devoted to the simplified model where the metal is an ideal one and dielectric is described by a constant dielectric permittivity. In the framework of this model a perturbation formalism applicable at low temperatures (short separations) is developed. In Section 4

the realistic case is considered when the dielectric permittivities of both metal and dielectric plates depend on the frequency. The analytic asymptotic expressions for the free energy, entropy and pressure of the Casimir interaction at both low and high temperatures are obtained. Section 5 contains the comparison between the analytical results and numerical computations using the tabulated optical data for plate materials. The application region of the derived asymptotic expressions is determined. In Section 6 it is shown that the inclusion of the dc conductivity in the description of dielectric plate leads to a violation of the Nernst heat theorem. Section 7 contains our conclusions and discussion.

2. Lifshitz formula in the configuration of metal and dielectric plates

We consider two thick parallel plates (semispaces) at temperature T in thermal equilibrium separated by the empty gap of width a . One plate is made of metal with the dielectric permittivity $\varepsilon^M(\omega)$ and another of dielectric with permittivity $\varepsilon^D(\omega)$. The free energy of the van der Waals and Casimir interaction between the plates per unit area is given by the Lifshitz formula [10,11,12,45,50]

$$\begin{aligned} \mathcal{F}(a, T) = & \frac{k_B T}{2\pi} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2} \delta_{0l}\right) \int_0^{\infty} k_{\perp} dk_{\perp} \\ & \times \left\{ \ln \left[1 - r_{\parallel}^M(\xi_l, k_{\perp}) r_{\parallel}^D(\xi_l, k_{\perp}) e^{-2a q_l} \right] \right. \\ & \left. + \ln \left[1 - r_{\perp}^M(\xi_l, k_{\perp}) r_{\perp}^D(\xi_l, k_{\perp}) e^{-2a q_l} \right] \right\}. \end{aligned} \quad (1)$$

Here the plates are perpendicular to the z axis, $k_{\perp} = |\mathbf{k}_{\perp}|$ is the magnitude of the wave vector in the plane of plates, $\xi_l = 2\pi k_B T l / \hbar$ are the Matsubara frequencies, and k_B is the Boltzmann constant. $r_{\parallel, \perp}^{M, D}$ are the reflection coefficients for metal (M) and dielectric (D) plates for the two independent polarizations of electromagnetic field calculated along the imaginary frequency axis. Index \parallel stands for the electric field parallel to the plane formed by \mathbf{k}_{\perp} and the z axis (transverse magnetic field), and index \perp stands for the electric field perpendicular to this plane (transverse electric field). The explicit expressions for the reflection coefficients are [45,50]

$$r_{\parallel}^{M, D}(\xi_l, k_{\perp}) = \frac{\varepsilon_l^{M, D} q_l - k_l^{M, D}}{\varepsilon_l^{M, D} q_l + k_l^{M, D}}, \quad r_{\perp}^{M, D}(\xi_l, k_{\perp}) = \frac{k_l^{M, D} - q_l}{k_l^{M, D} + q_l}, \quad (2)$$

where

$$q_l = \sqrt{\frac{\xi_l^2}{c^2} + k_\perp^2}, \quad k_l^{M,D} = \sqrt{\varepsilon_l^{M,D} \frac{\xi_l^2}{c^2} + k_\perp^2}, \quad (3)$$

and

$$\varepsilon_l^{M,D} = \varepsilon^{M,D}(i\xi_l). \quad (4)$$

The pressure of the van der Waals and Casimir interaction between metal and dielectric (i.e., the force per unit area of plates) is obtained from

$$P(a, T) = -\frac{\partial \mathcal{F}(a, T)}{\partial a}. \quad (5)$$

Using Eq. (1) we arrive at

$$P(a, T) = -\frac{k_B T}{\pi} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2} \delta_{0l}\right) \int_0^\infty k_\perp dk_\perp q_l \times \left[\frac{r_\parallel^M(\xi_l, k_\perp) r_\parallel^D(\xi_l, k_\perp)}{e^{2aq_l} - r_\parallel^M(\xi_l, k_\perp) r_\parallel^D(\xi_l, k_\perp)} + \frac{r_\perp^M(\xi_l, k_\perp) r_\perp^D(\xi_l, k_\perp)}{e^{2aq_l} - r_\perp^M(\xi_l, k_\perp) r_\perp^D(\xi_l, k_\perp)} \right]. \quad (6)$$

Using the proximity force theorem [51], one can obtain from Eq. (1) the approximate expression for the Casimir force acting between a sphere and a plate

$$F(a, T) = 2\pi R \mathcal{F}(a, T). \quad (7)$$

This equation is widely used for the interpretation of measurements of the Casimir force [13,14,15,16,17,18,19,20,21,22,23,24,25,26]. Recently both exact analytic and numerical results for the Casimir force in the configuration of a cylinder above a plate (electromagnetic case) and for a sphere above a plate (scalar case) were obtained [52,53,54,55]. It was shown that the error introduced by the use of Eq. (7) for the experimental parameters in already performed experiments is less than 0.1%. Using Eq. (7), the analytical results derived below for metal and dielectric plates, can be immediately applied for the interpretation of measurements of the Casimir force between Au coated sphere and Si plate [25,26,27,28].

The analytic perturbation expansions in Eqs. (1) and (6) can be conveniently performed by using the dimensionless variables ζ_l and y

$$\zeta_l = \frac{\xi_l}{\omega_c} = \frac{2a\xi_l}{c} = \tau l, \quad y = 2aq_l, \quad (8)$$

where $\omega_c = c/(2a)$ is the characteristic frequency of the Casimir effect and $\tau = 4\pi k_B a T / (\hbar c)$. In terms of these variables the free energy (1) takes the form

$$\begin{aligned} \mathcal{F}(a, T) = \frac{\hbar c \tau}{32\pi^2 a^3} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\delta_{0l}\right) \int_{\zeta_l}^{\infty} y dy \left\{ \ln \left[1 - r_{\parallel}^M(\zeta_l, y) r_{\parallel}^D(\zeta_l, y) e^{-y}\right] \right. \\ \left. + \ln \left[1 - r_{\perp}^M(\zeta_l, y) r_{\perp}^D(\zeta_l, y) e^{-y}\right] \right\}. \end{aligned} \quad (9)$$

Using the variables (8), the reflection coefficients (2) are

$$\begin{aligned} r_{\parallel}^{M,D}(\zeta_l, y) &= \frac{\varepsilon_l^{M,D} y - \sqrt{y^2 + \zeta_l^2(\varepsilon_l^{M,D} - 1)}}{\varepsilon_l^{M,D} y + \sqrt{y^2 + \zeta_l^2(\varepsilon_l^{M,D} - 1)}}, \\ r_{\perp}^{M,D}(\zeta_l, y) &= \frac{\sqrt{y^2 + \zeta_l^2(\varepsilon_l^{M,D} - 1)} - y}{\sqrt{y^2 + \zeta_l^2(\varepsilon_l^{M,D} - 1)} + y}, \end{aligned} \quad (10)$$

where in accordance with Eq. (4) $\varepsilon_l^{M,D} = \varepsilon^{M,D}(i\zeta_l\omega_c)$.

The pressure (6) is rearranged as follows:

$$\begin{aligned} P(a, T) = -\frac{\hbar c \tau}{32\pi^2 a^4} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\delta_{0l}\right) \int_{\zeta_l}^{\infty} y^2 dy \\ \times \left[\frac{r_{\parallel}^M(\zeta_l, y) r_{\parallel}^D(\zeta_l, y)}{e^y - r_{\parallel}^M(\zeta_l, y) r_{\parallel}^D(\zeta_l, y)} + \frac{r_{\perp}^M(\zeta_l, y) r_{\perp}^D(\zeta_l, y)}{e^y - r_{\perp}^M(\zeta_l, y) r_{\perp}^D(\zeta_l, y)} \right]. \end{aligned} \quad (11)$$

The other important characteristic of the van der Waals and Casimir interaction is the entropy

$$S(a, T) = -\frac{\partial \mathcal{F}(a, T)}{\partial T}. \quad (12)$$

In [32,33] the behavior of the Casimir entropy at $T \rightarrow 0$ was used as a phenomenological constraint on the selection of theoretically consistent models of

the dielectric response for real metals at low frequencies. It was proposed that all consistent models should satisfy the thermodynamic condition $S(a, 0) = 0$, i.e., be in agreement with the Nernst heat theorem. In [45] it was demonstrated that this condition is respected for two dielectric plates with the finite static dielectric permittivities. The new analytic expressions for the free energy obtained in the present paper permit investigate the behavior of entropy in the configuration of one metal and one dielectric plate and find when it vanishes with vanishing temperature.

3. Model of ideal metal and dielectric with constant dielectric permittivity

To find the analytic expressions for the free energy, pressure and entropy of the Casimir interaction between metal and dielectric, we start from a simplified model when the metal is an ideal one and the dielectric possesses some finite dielectric permittivity ε_0^D independent on the frequency. Such modeling is widely used in Casimir physics (see, e.g., [2,3,5,6,8,10,11,12,56]). It provides rather good description of real metals and dielectrics at sufficiently large separations between the interacting surfaces. For an ideal metal it holds $|\varepsilon^M| = \infty$ at all frequencies and from Eq. (10) one obtains

$$r_{\parallel}^M(\zeta_l, y) = 1, \quad r_{\perp}^M(\zeta_l, y) = 1, \quad l \geq 0. \quad (13)$$

Using Eq. (13), the free energy (9) and pressure (11) are represented in a more simple form,

$$\begin{aligned} \mathcal{F}(a, T) &= \frac{\hbar c \tau}{32\pi^2 a^3} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\delta_{0l}\right) \int_{\zeta_l}^{\infty} y dy \\ &\quad \times \left\{ \ln \left[1 - r_{\parallel}^D(\zeta_l, y)e^{-y}\right] + \ln \left[1 - r_{\perp}^D(\zeta_l, y)e^{-y}\right] \right\}, \\ P(a, T) &= -\frac{\hbar c \tau}{32\pi^2 a^4} \sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\delta_{0l}\right) \int_{\zeta_l}^{\infty} y^2 dy \\ &\quad \times \left[\frac{r_{\parallel}^D(\zeta_l, y)}{e^y - r_{\parallel}^D(\zeta_l, y)} + \frac{r_{\perp}^D(\zeta_l, y)}{e^y - r_{\perp}^D(\zeta_l, y)} \right]. \end{aligned} \quad (14)$$

Notice that in the framework of our model in Eq. (10) it holds $\varepsilon_l^D = \varepsilon_0^D$, i.e., the dielectric permittivities computed at different imaginary Matzubara

frequencies do not depend on l . In particular, at $l = 0$ it follows:

$$r_{\parallel}^D(0, y) = \frac{\varepsilon_0^D - 1}{\varepsilon_0^D + 1} \equiv r_0, \quad r_{\perp}^D(0, y) = 0. \quad (15)$$

In fact Eq. (15) is valid not only for our simplified model but for any dielectric with a finite static dielectric permittivity $\varepsilon^D(0) \equiv \varepsilon_0^D < \infty$. Usually for nonpolar dielectrics $\varepsilon^D(i\xi) = \varepsilon_0^D = \text{const}$ in the frequency region from $\xi = 0$ up to rather high frequencies of about 10^{15} rad/s and for higher frequencies $\varepsilon^D(i\xi)$ decreases to unity. The simplified model does not take the latter into account (in the next section we show that this does not influence the first terms in the asymptotic behavior of the free energy, entropy and pressure at low temperature).

Now we proceed with the derivation of the asymptotic behavior of the Casimir free energy at low temperature ($\tau \ll 1$). Using the Abel-Plana formula [3,8]

$$\sum_{l=0}^{\infty} \left(1 - \frac{1}{2}\delta_{l0}\right) F(l) = \int_0^{\infty} F(t) dt + i \int_0^{\infty} dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1}, \quad (16)$$

where $F(z)$ is an analytic function in the right-plane, we can rearrange Eq. (14) to the form

$$\mathcal{F}(a, T) = E(a) + \Delta\mathcal{F}(a, T). \quad (17)$$

Here,

$$E(a) = \frac{\hbar c}{32\pi^2 a^3} \int_0^{\infty} d\zeta \int_{\zeta}^{\infty} f(\zeta, y) dy \quad (18)$$

is the energy of the Casimir interaction at zero temperature,

$$\Delta\mathcal{F}(a, T) = \frac{i\hbar c\tau}{32\pi^2 a^3} \int_0^{\infty} dt \frac{F(i\tau t) - F(-i\tau t)}{e^{2\pi t} - 1} \quad (19)$$

is the thermal correction to it, and the following notations are introduced,

$$f(\zeta, y) = y \ln [1 - r_{\parallel}^D(\zeta, y)e^{-y}] + y \ln [1 - r_{\perp}^D(\zeta, y)e^{-y}],$$

$$F(x) = \int_x^{\infty} dy f(x, y). \quad (20)$$

The expansion of $f(x, y)$ in powers of x takes the form

$$f(x, y) = y \ln(1 - r_0 e^{-y}) - x^2 \left(\frac{\varepsilon_0^D - 1}{4y} e^{-y} - \frac{\varepsilon_0^D}{\varepsilon_0^D + 1} \sum_{n=1}^{\infty} r_0^n \frac{e^{-ny}}{y} \right) + O(x^3). \quad (21)$$

To find $F(x)$ in Eq. (20) we integrate the right-hand side of Eq. (21) with respect to y . Notice that the first term on the right-hand side of Eq. (21) does not contribute to the first expansion orders of $F(ix) - F(-ix)$ which is in fact the quantity of our interest. This is because in the expression

$$\int_x^\infty y dy \ln(1 - r_0 e^{-y}) = \int_0^\infty v dv \ln(1 - r_0 e^{-v}) + O(x^2), \quad (22)$$

where the new variable $v = y - x$ was introduced, the first-order in x contribution vanishes. Thus, this term could contribute to $F(ix) - F(-ix)$ only starting from the third expansion order. Integrating the second term on the right-hand side of Eq. (21) using the formula

$$\int_x^\infty dy \frac{e^{-ny}}{y} = -\text{Ei}(-nx), \quad (23)$$

where $\text{Ei}(z)$ is the exponential integral function, we finally obtain

$$F(ix) - F(-ix) = i\pi \frac{(\varepsilon_0^D - 1)^2}{4(\varepsilon_0^D + 1)} x^2 - i\gamma x^3 + O(x^4), \quad (24)$$

where the third order real expansion coefficient γ cannot be determined at this stage of our calculations because all powers in the expansion of $f(x, y)$ in powers of x contribute to it.

Now we substitute Eq. (24) in Eq. (19) and find the free energy (17)

$$\mathcal{F}(a, T) = E(a) - \frac{\hbar c}{32\pi^2 a^3} \left[\frac{\zeta(3)}{16\pi^2} \frac{(\varepsilon_0^D - 1)^2}{\varepsilon_0^D + 1} \tau^3 - K_4 \tau^4 + O(\tau^5) \right], \quad (25)$$

where $K_4 = \gamma/240$ and $\zeta(z)$ is the Riemann zeta function.

The Casimir pressure is obtained from Eqs. (5) and (25). It is equal to

$$P(a, T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} \left[K_4 \tau^4 + O(\tau^5) \right], \quad (26)$$

where $P_0(a) = -\partial E(a)/\partial a$.

In order to determine the coefficients K_4 , we now start from the Lifshitz representation of the pressure in Eq. (14). Using the Abel-Plana formula (16), we rearrange Eq. (14) to the form analogical to (17)–(19),

$$\begin{aligned} P(a, T) &= P_0(a) + \Delta P(a, T), \\ P_0(a) &= \frac{3\hbar c}{32\pi^2 a^4} \int_0^\infty d\zeta \int_\zeta^\infty f(\zeta, y) dy, \\ \Delta P(a, T) &= -\frac{i\hbar c \tau}{32\pi^2 a^4} \int_0^\infty dt \frac{\Phi(i\tau t) - \Phi(-i\tau t)}{e^{2\pi t} - 1}. \end{aligned} \quad (27)$$

Here $P_0(a)$ is the Casimir pressure at zero temperature, $\Delta P(a, T)$ is the thermal correction to it and the following notation is introduced:

$$\Phi_{\parallel, \perp}(x) = \int_x^\infty dy \frac{y^2 r_{\parallel, \perp}(x, y)}{e^y - r_{\parallel, \perp}(x, y)}. \quad (28)$$

To find the expansion of $\Phi(ix) - \Phi(-ix)$ in powers of x , we first deal with $\Phi_\perp(x)$. By adding and subtracting the asymptotic behavior of the intergrand function at small x ,

$$\frac{y^2 r_\perp(x, y)}{e^y - r_\perp(x, y)} = \frac{1}{4}(\varepsilon_0^D - 1)x^2 e^{-y} + O(x^3), \quad (29)$$

under the integral in Eq. (28) and introducing the new variable $v = y/x$, the function $\Phi_\perp(x)$ can be identically rearranged and expanded in powers of x as follows:

$$\begin{aligned} \Phi_\perp(x) &= \frac{1}{4}(\varepsilon_0^D - 1)x^2 e^{-x} + x^3 \int_1^\infty dv \left[v^2 \sum_{n=1}^\infty r_\perp^n(v) e^{-nvx} - \frac{1}{4}(\varepsilon_0^D - 1)e^{-vx} \right] \\ &= \frac{1}{4}(\varepsilon_0^D - 1)x^2(1 - x) \\ &\quad + x^3 \int_1^\infty dv \left[\frac{v^2 r_\perp(v)}{1 - r_\perp(v)} - \frac{\varepsilon_0^D - 1}{4} \right] + O(x^4). \end{aligned} \quad (30)$$

Calculating the integral on the right-hand side of Eq. (30), we arrive at the result

$$\Phi_\perp(x) = \frac{\varepsilon_0^D - 1}{4}x^2 - \frac{1}{6} \left(\varepsilon_0^D \sqrt{\varepsilon_0^D} - 1 \right) x^3 + O(x^4). \quad (31)$$

To deal with $\Phi_{\parallel}(x)$, we add and subtract under the integral in Eq. (28) the two first expansion terms of the integrated function in powers of x ,

$$\begin{aligned}\Phi_{\parallel}(x) = & \int_x^{\infty} y^2 dy \left[\frac{r_0}{e^y - r_0} - \frac{\varepsilon_0^D r_0 e^{-y} x^2}{y^2 (\varepsilon_0^D + 1) (1 - r_0 e^{-y})^2} \right] \\ & + \int_x^{\infty} y^2 dy \left[\frac{r_{\parallel}(x, y)}{e^y - r_{\parallel}(x, y)} - \frac{r_0}{e^y - r_0} + \frac{\varepsilon_0^D r_0 e^{-y} x^2}{y^2 (\varepsilon_0^D + 1) (1 - r_0 e^{-y})^2} \right].\end{aligned}\quad (32)$$

The asymptotic expansions of the first and second integrals on the right-hand side of Eq. (32) are

$$2\text{Li}_3(r_0) - \frac{\varepsilon_0^D(\varepsilon_0^D - 1)}{2(\varepsilon_0^D + 1)}x^2 + \frac{1}{12}(\varepsilon_0^D - 1)(3\varepsilon_0^D - 2)x^3 + \text{O}(x^4), \quad (33)$$

$$\begin{aligned}\left[-\frac{1}{4}\varepsilon_0^D(\varepsilon_0^D - 1) - \frac{1}{6}\varepsilon_0^D(\varepsilon_0^D \sqrt{\varepsilon_0^D} - 1) + \frac{1}{2}\varepsilon_0^D(\varepsilon_0^D - 1)\sqrt{\varepsilon_0^D} \right] x^3 \\ + \text{O}(x^4),\end{aligned}\quad (34)$$

respectively, where $\text{Li}_n(z)$ is the polylogarithm function. By summing Eqs. (33) and (34) we find

$$\begin{aligned}\Phi_{\parallel}(x) = & 2\text{Li}_3(r_0) - \frac{\varepsilon_0^D(\varepsilon_0^D - 1)}{2(\varepsilon_0^D + 1)}x^2 \\ & - \frac{1}{6} \left[(\varepsilon_0^D - 1) + (\varepsilon_0^D \sqrt{\varepsilon_0^D} - 1) - 3\varepsilon_0^D(\varepsilon_0^D - 1)\sqrt{\varepsilon_0^D} \right] x^3 + \text{O}(x^4).\end{aligned}\quad (35)$$

After summing Eqs. (31) and (35) the following result is obtained:

$$\Phi(ix) - \Phi(-ix) = -\frac{2i}{3} \left(1 - 2\varepsilon_0^D \sqrt{\varepsilon_0^D} + (\varepsilon_0^D)^2 \sqrt{\varepsilon_0^D} \right) x^3 + \text{O}(x^4). \quad (36)$$

Substituting this in Eq. (27) and integrating we arrive at the asymptotic expression for the Casimir pressure in the limit of small τ ,

$$P(a, T) = P_0(a) - \frac{\hbar c}{32\pi^2 a^4} \left[\frac{1 - 2\varepsilon_0^D \sqrt{\varepsilon_0^D} + (\varepsilon_0^D)^2 \sqrt{\varepsilon_0^D}}{360} \tau^4 + \text{O}(\tau^5) \right]. \quad (37)$$

The comparison of this equation with Eq. (26) leads to the explicit expression for the coefficient K_4 :

$$K_4 = \frac{1}{360} \left(1 - 2\varepsilon_0^D \sqrt{\varepsilon_0^D} + (\varepsilon_0^D)^2 \sqrt{\varepsilon_0^D} \right) \quad (38)$$

and, thus, the explicit asymptotic expression (25) for the free energy is also fully determined.

Notice that the energy and pressure at zero temperature [$E(a)$ and $P_0(a)$ defined in Eqs. (18) and (27), respectively] depend on the separation through only the factors a^{-3} and a^{-4} in front of the integrals. They can be conveniently presented in the form

$$\begin{aligned} E(a) &= -\frac{\pi^2}{720} \frac{\hbar c}{a^3} \psi_{DM}(\varepsilon_0^D), \\ P_0(a) &= -\frac{\pi^2}{240} \frac{\hbar c}{a^4} \psi_{DM}(\varepsilon_0^D), \end{aligned} \quad (39)$$

where the function $\psi_{DM}(\varepsilon_0)$ is defined as

$$\psi_{DM}(\varepsilon_0^D) = -\frac{45}{2\pi^4} \int_0^\infty d\zeta \int_\zeta^\infty f(\zeta, y) dy. \quad (40)$$

In fact, ψ_{DM} in Eqs. (39), (40) is the correction factor to the famous Casimir result [1] obtained for two ideal metals. It is equal to the function φ_{DM} introduced in [12], multiplied by r_0 .

The function $\psi_{DM}(\varepsilon_0^D)$ in Eq. (40) can be presented in a more simple analytical form as follows. Presenting the logarithms in Eq. (20) as series and changing the order of integrations, one obtains

$$\begin{aligned} \psi_{DM}(\varepsilon_0^D) &= \frac{45}{2\pi^4} \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty y dy e^{-ny} \\ &\times \int_0^y d\zeta \left\{ \left[r_\parallel^D(\zeta, y) \right]^n + \left[r_\perp^D(\zeta, y) \right]^n \right\}. \end{aligned} \quad (41)$$

Introducing the new variable $w = \zeta/y$, we rearrange Eq. (41) to the form

$$\begin{aligned} \psi_{DM}(\varepsilon_0^D) &= \frac{45}{2\pi^4} \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty y^2 dy e^{-ny} \\ &\times \int_0^1 dw \left\{ \left[r_\parallel^D(w) \right]^n + \left[r_\perp^D(w) \right]^n \right\}, \end{aligned} \quad (42)$$

where

$$\begin{aligned}
r_{\parallel}^D(w) &= \frac{\varepsilon_0^D - \sqrt{1 + (\varepsilon_0^D - 1)w^2}}{\varepsilon_0^D + \sqrt{1 + (\varepsilon_0^D - 1)w^2}}, \\
r_{\perp}^D(w) &= \frac{\sqrt{1 + (\varepsilon_0^D - 1)w^2} - 1}{\sqrt{1 + (\varepsilon_0^D - 1)w^2} + 1}.
\end{aligned} \tag{43}$$

Calculating the integral in y and performing the summation with respect to n in Eq. (42) one arrives at

$$\psi_{DM}(\varepsilon_0^D) = \frac{45}{\pi^4} \int_0^{\infty} dw \left\{ \text{Li}_4 \left[r_{\parallel}^D(w) \right] + \text{Li}_4 \left[r_{\perp}^D(w) \right] \right\}. \tag{44}$$

In Fig. 1 the function (44) is plotted versus ε_0^D as a solid line (when $\varepsilon_0^D \rightarrow 1$ it goes to zero and when $\varepsilon_0^D \rightarrow \infty$ it goes to unity reproducing the limit of ideal metals).

It is notable that the model under consideration represents correctly the Casimir energy and pressure (39) at $T = 0$ in only the retarded regime (i.e., at sufficiently large separations). As to the thermal corrections in Eqs. (25) and (37), the obtained expressions are valid also at short separations in a nonretarded regime under the condition that the parameter τ is sufficiently small due to sufficiently low temperature.

From Eqs. (12) and (25) the asymptotic behavior of the Casimir entropy in the limit of small τ is given by

$$\begin{aligned}
S(a, T) &= \frac{3k_B \zeta(3) (\varepsilon_0^D - 1)^2}{128\pi^3 a^2 (\varepsilon_0^D + 1)} \tau^2 \\
&\times \left[1 - \frac{8\pi^2 (\varepsilon_0^D + 1) \left(1 - 2\varepsilon_0^D \sqrt{\varepsilon_0^D} + (\varepsilon_0^D)^2 \sqrt{\varepsilon_0^D} \right)}{135\zeta(3) (\varepsilon_0^D - 1)^2} \tau + \text{O}(\tau^2) \right].
\end{aligned} \tag{45}$$

As is seen from Eq. (45), the entropy of the Casimir interaction between metal and dielectric plates vanishes with vanishing temperature as is required by the Nernst heat theorem (note that the first term of order τ^2 in Eq. (45) was obtained in [50]). The important property of the perturbation expansions in powers of τ in Eqs. (25), (37) and (45) is that it is impermissible to consider the limiting case $\varepsilon_0^D \rightarrow \infty$ in order to obtain the case of two ideal metals like it was discussed above in application to Eq. (39). The mathematical reason is that in the power expansion of functions depending on ε_0^D as a parameter the limiting transitions $\varepsilon_0^D \rightarrow \infty$ and $\tau \rightarrow 0$ are not interchangeable. Of great importance is the possibility to apply Eq. (45) at as small T as is wished. This

is the principal advantage of analytical calculations as compared to numerical ones.

Now we consider the opposite limiting case $\tau \gg 1$, i.e., the limit of high temperatures (large separations). Here the main contribution to the free energy (14) is given by the term with $l = 0$ whereas all terms with $l \geq 1$ are exponentially small [8],

$$\mathcal{F}(a, T) = \frac{\hbar c \tau}{64\pi^2 a^3} \int_0^\infty y dy \ln(1 - r_0 e^{-y}). \quad (46)$$

By integrating in Eq. (46) we obtain

$$\mathcal{F}(a, T) = -\frac{k_B T}{16\pi a^2} \text{Li}_3(r_0). \quad (47)$$

For the Casimir pressure and entropy at $\tau \gg 1$ from Eqs. (5), (12) and (47) it follows

$$P(a, T) = -\frac{k_B T}{8\pi a^3} \text{Li}_3(r_0), \quad S(a, T) = \frac{k_B}{16\pi a^2} \text{Li}_3(r_0). \quad (48)$$

4. Thermal Casimir force between metal and dielectric with frequency-dependent dielectric permittivities

In this section we obtain the analytic expressions for the low-temperature behavior of the Casimir interaction between metal and dielectric plates taking into account the dependence of their dielectric permittivities on the frequency. The metal plate is described by the dielectric permittivity of the plasma model,

$$\varepsilon^M(i\xi_l) = 1 + \frac{\omega_p^2}{\xi_l^2}, \quad (49)$$

where $\omega_p = 2\pi c/\lambda_p$ is the plasma frequency, and λ_p is the plasma wavelength. In the theory of the thermal Casimir force this description was first used in [37,38] and was shown to work good at separations between plates greater than the plasma wavelength. At such separations the characteristic frequency of the Casimir effect ω_c belongs to the region of infrared optics where the relaxation processes do not play any role [57].

For dielectric plate we use the Ninham-Parsegian representation of the dielectric permittivity along the imaginary frequency axis [58,59],

$$\varepsilon^D(i\xi_l) = 1 + \sum_j \frac{C_j}{1 + \frac{\xi_l^2}{\omega_j^2}}. \quad (50)$$

Here C_j are the absorption strengths satisfying the condition

$$\sum_j C_j = \varepsilon_0^D - 1, \quad (51)$$

and ω_j are the characteristic absorption frequencies [recall that now $\varepsilon_0^D = \varepsilon^D(0) < \infty$]. Eq. (50) gives a very accurate approximate description of the dielectric properties for many dielectrics. It has been successfully used by many authors for the comparison of experimental data with theory [60].

From Eq. (50) we return to the same values (15) of the reflection coefficients of the dielectric plate at zero frequency as were obtained in the simplified model of the frequency-independent dielectric permittivity. Thus, due to the zero value of $r_\perp^D(0, y)$, the transverse electric mode at zero frequency does not contribute to the free energy (9) of the Casimir interaction between metal and dielectric regardless of the value of $r_\perp^M(0, y)$ for a metal. As was told in the Introduction, there are different approaches on how to correctly calculate the transverse electric coefficient at zero frequency, $r_\perp^M(0, y)$, for a plate made of real metal. In the configuration of metal and dielectric this problem, however, does not influence the result. Note that if we would use instead of Eq. (49) the Drude model, taking relaxation into account, the prime perturbation orders in all results below remain unchanged for metals with perfect crystal lattices. The role of impurities in the validity of the Nernst heat theorem in the case of two metal plates is discussed in [31,33,41,42,43,44,61].

We start from Eq. (9) for the free energy. Once again, using the Abel-Plana formula, Eq. (9) can be represented by Eq. (17) as the sum of $\hat{E}(a)$ in Eq. (18) and $\Delta\hat{\mathcal{F}}(a, T)$ in Eqs. (19) and Eq. (20), where we mark by a hat all quantities related to *real* metal and dielectric. The single difference is that the function $f(x, y)$ in Eq. (20) should be replaced by

$$\begin{aligned} \hat{f}(x, y) &\equiv \hat{f}_\parallel(x, y) + \hat{f}_\perp(x, y), \\ \hat{f}_{\parallel,\perp}(x, y) &= y \ln \left[1 - \hat{r}_{\parallel,\perp}^M(x, y) \hat{r}_{\parallel,\perp}^D(x, y) e^{-y} \right]. \end{aligned} \quad (52)$$

It is notable that for real metal and dielectric $\hat{E}(a)$ and $\Delta\hat{\mathcal{F}}(a, T)$ in Eq. (17) may lose the obvious meaning of the energy at zero temperature and the thermal correction to it. In fact, this meaning is preserved only in the case

when the dielectric permittivities $\varepsilon^{M,D}(i\xi)$ do not depend on the temperature as a parameter like it was in Section 3. In the latter case it holds

$$\Delta\mathcal{F}(a, T) = \mathcal{F}(a, T) - \mathcal{F}(a, 0) = \mathcal{F}(a, T) - E(a) \quad (53)$$

in accordance with the intuitive definition of the thermal correction. If, however, $\varepsilon^M(i\xi)$ or $\varepsilon^D(i\xi)$ or both depend on the temperature as a parameter, Eq. (53) is violated. In this case $\Delta\hat{\mathcal{F}}(a, T)$ defined in Eq. (19) takes into account only the part of temperature dependence of the free energy originating from the Matsubara frequencies and is not equal to $\hat{\mathcal{F}}(a, T) - \hat{\mathcal{F}}(a, 0)$. Moreover, in this case $\hat{E}(a)$ in Eqs. (17) and (18) is in fact temperature-dependent and it would be more correct to use the notation $\hat{E} = \hat{E}(a, T)$.

To obtain the analytic expressions of our interest we develop the perturbation theory in two small parameters τ and $\eta \equiv \delta/(2a)$, where $\delta = \lambda_p/(2\pi)$ is the penetration depth of the electromagnetic oscillations into a metal. For the sake of simplicity we will consider dielectrics which can be described by Eq. (50) with only one oscillator, i.e., with $j = 1$. The high-resistivity Si is a typical example of such materials. The function $F(x)$ in Eq. (20) can be conveniently presented in the form

$$\hat{F}(x) = \hat{F}_{\parallel}(x) + \hat{F}_{\perp}(x), \quad \hat{F}_{\parallel,\perp}(x) = \int_x^{\infty} dy \hat{f}_{\parallel,\perp}(x, y). \quad (54)$$

As a first step we perform the expansion with respect to the powers of small parameter η . This results in:

$$\begin{aligned} \hat{F}_{\parallel}(x) &= \int_x^{\infty} y dy \ln [1 - \hat{r}_{\parallel}^D(x, y)e^{-y}] + 2x^2\eta \int_x^{\infty} dy \frac{\hat{r}_{\parallel}^D(x, y)}{e^y - \hat{r}_{\parallel}^D(x, y)} \\ &\quad - 2x^4\eta^2 \int_x^{\infty} dy \frac{e^y \hat{r}_{\parallel}^D(x, y)}{y[e^y - \hat{r}_{\parallel}^D(x, y)]^2} + O(\eta^3), \\ \hat{F}_{\perp}(x) &= \int_x^{\infty} y dy \ln [1 - \hat{r}_{\perp}^D(x, y)e^{-y}] + 2\eta \int_x^{\infty} y^2 dy \frac{\hat{r}_{\perp}^D(x, y)}{e^y - \hat{r}_{\perp}^D(x, y)} \\ &\quad - 2\eta^2 \int_x^{\infty} y^3 dy \frac{e^y \hat{r}_{\perp}^D(x, y)}{[e^y - \hat{r}_{\perp}^D(x, y)]^2} + O(\eta^3). \end{aligned} \quad (55)$$

The dielectric reflection coefficients in Eq. (55) are obtained after the substitution of Eqs. (50) and (51) with $j = 1$ in Eq. (10):

$$\begin{aligned}
\hat{r}_{\parallel}^D(x, y) &= \frac{\left(1 + \frac{\varepsilon_0^D - 1}{1 + b^2 x^2}\right) y - \sqrt{y^2 + \frac{\varepsilon_0^D - 1}{1 + b^2 x^2} x^2}}{\left(1 + \frac{\varepsilon_0^D - 1}{1 + b^2 x^2}\right) y + \sqrt{y^2 + \frac{\varepsilon_0^D - 1}{1 + b^2 x^2} x^2}}, \\
\hat{r}_{\perp}^D(x, y) &= \frac{\sqrt{y^2 + \frac{\varepsilon_0^D - 1}{1 + b^2 x^2} x^2} - y}{\sqrt{y^2 + \frac{\varepsilon_0^D - 1}{1 + b^2 x^2} x^2} + y},
\end{aligned} \tag{56}$$

where $b \equiv \omega_c/\omega_1$.

Let us consecutively consider the contributions to $\hat{F}(x)$ from the terms of order η^0 , η and η^2 in Eq. (55). As to the terms of order η^0 [the first and fourth lines in Eq. (55)], the expansion in powers of small x (small τ) performed using Eq. (56) leads to

$$\begin{aligned}
\hat{F}_{\eta^0}(x) &= F(x) - b^2 x^4 \left\{ \frac{3(\varepsilon_0^D)^2 + 2\varepsilon_0^D - 1}{(\varepsilon_0^D + 1)^2} \sum_{n=1}^{\infty} n r_0^n \text{Ei}(-nx) \right. \\
&\quad \left. - r_0^2 \sum_{n=1}^{\infty} n r_0^n \text{Ei}[-(n+1)x] + \frac{\varepsilon_0^D - 1}{4} \text{Ei}(-x) \right\} + \text{O}(x^5).
\end{aligned} \tag{57}$$

Here $F(x)$ was already calculated in Sec. III and results in Eq. (24). The additional contributions to the right-hand side of Eq. (57) lead to the term of order τ^4 in $\hat{F}_{\eta^0}(i\tau t) - \hat{F}_{\eta^0}(-i\tau t)$ and of order τ^5 in the free energy. Thus, they can be omitted (like in Sec. III, we preserve only the terms of order τ^3 and τ^4).

To find the contribution to $\hat{F}(x)$ of order η [we use the notation $\hat{F}_{\eta}(x)$], we expand in powers of x the following quantities under the integrals in Eq. (55):

$$\begin{aligned}
\frac{x^2 \hat{r}_{\parallel}^D(x, y)}{e^y - \hat{r}_{\parallel}^D(x, y)} &= x^2 \frac{r_0}{e^y - r_0} - x^4 \frac{e^y r_0 \varepsilon_0^D}{y^2 (\varepsilon_0^D + 1) (e^y - r_0)^2} \\
&\quad - 2b^2 x^4 \frac{r_0 e^y}{(\varepsilon_0^D + 1) (e^y - r_0)^2} + \text{O}(x^5), \\
\frac{y^2 \hat{r}_{\perp}^D(x, y)}{e^y - \hat{r}_{\perp}^D(x, y)} &= x^2 \frac{(\varepsilon_0^D - 1) e^{-y}}{4} - x^4 \frac{(\varepsilon_0^D - 1)^2 e^{-2y} (2e^y - 1)}{16y^2} \\
&\quad - b^2 x^4 \frac{(\varepsilon_0^D - 1) e^{-y}}{4} + \text{O}(x^5).
\end{aligned} \tag{58}$$

By integrating of the third terms on the right-hand side of equations (58) with respect to y from x to infinity, we find that they contribute to $\hat{F}_{\eta}(i\tau t) - \hat{F}_{\eta}(-i\tau t)$ only in the order τ^5 and, thus, to the free energy in the order τ^6 .

Because of this they can be omitted. The integration of the first two terms on the right-hand side of equations (58) leads to

$$\begin{aligned}\hat{F}_{\eta,\parallel}(i\tau t) - \hat{F}_{\eta,\parallel}(-i\tau t) &= i\eta\tau^3 t^3 (\varepsilon_0^D - 1)(\varepsilon_0^D + 2), \\ \hat{F}_{\eta,\perp}(i\tau t) - \hat{F}_{\eta,\perp}(-i\tau t) &= i\eta\tau^3 t^3 \frac{(\varepsilon_0^D - 1)(\varepsilon_0^D + 3)}{4}.\end{aligned}\tag{59}$$

From Eq. (59) it follows

$$\hat{F}_\eta(i\tau t) - \hat{F}_\eta(-i\tau t) = i\eta\tau^3 t^3 \frac{(\varepsilon_0^D - 1)(5\varepsilon_0^D + 11)}{4}.\tag{60}$$

As to the terms of order η^2 in Eq. (55), their lowest order contributions to $\hat{F}_{\eta^2,\parallel}(i\tau t) - \hat{F}_{\eta^2,\parallel}(-i\tau t)$ and to $\hat{F}_{\eta^2,\perp}(i\tau t) - \hat{F}_{\eta^2,\perp}(-i\tau t)$ are of order τ^4 and τ^5 , respectively. This leads to the respective contributions of order τ^5 and τ^6 to the free energy which we omit in our analysis.

Using Eq. (19), the respective correction to the Casimir free energy takes the form

$$\Delta\hat{\mathcal{F}}_\eta(a, T) = -\frac{\hbar c}{30720\pi^2 a^3} \eta\tau^4 (\varepsilon_0^D - 1)(5\varepsilon_0^D + 11).\tag{61}$$

Remarkably, $\eta\tau^4 \sim a^3$ and the correction (61) does not depend on the separation. Thus, there is no correction to the Casimir pressure of order $\eta\tau^q$ with $q \leq 4$ due to the finite conductivity of a metal plate. Recall that in the configuration of two ideal metal plates the main thermal correction at low temperature is of order τ^4 . If the nonideality of a metal is taken into account, the correction of order τ^3 arises [8]. From this it follows that the thermal correction in the configuration metal-dielectric is less sensitive to the finite conductivity of a metal than in configuration of two metals.

Combining the contributions from the zeroth and first orders in η in Eqs. (25) and (61), the free energy at low temperatures for the configuration or real metal and real dielectric is

$$\begin{aligned}\hat{\mathcal{F}}(a, T) &= \hat{E}(a) - \frac{\hbar c}{32\pi^2 a^3} \left[\zeta(3) \frac{(\varepsilon_0^D - 1)^2}{16\pi^2 \varepsilon_0^D + 1} \tau^3 - K_4 \tau^4 \right. \\ &\quad \left. + \frac{1}{960} (\varepsilon_0^D - 1)(5\varepsilon_0^D + 11) \eta\tau^4 + O(\tau^5) \right],\end{aligned}\tag{62}$$

where K_4 is defined in Eq. (38). It is notable that the low-temperature behavior of the free energy is not influenced by the absorption bands of the dielectric material and are determined by only the static dielectric permittivity. This is

in analogy to the case of two dielectric plates [45,46]. For the Casimir pressure between plates made of real metal and dielectric Eq. (37) is preserved with the replacement of $P_0(a)$ for $\hat{P}_0(a)$ given below.

Now we derive the analytic representation for the Casimir energy $\hat{E}(a)$ in the configuration with one plate made of real metal and another plate made of real dielectric. Expanding in powers of η in Eq. (18), with f replaced by \hat{f} from Eq. (52), we obtain

$$\begin{aligned} \hat{E}(a) = \frac{\hbar c}{32\pi^2 a^3} & \left\{ \int_0^\infty d\zeta \int_\zeta^\infty y dy \left[\ln \left(1 - \hat{r}_\parallel^D(\zeta, y) e^{-y} \right) \right. \right. \\ & + \ln \left(1 - \hat{r}_\perp^D(\zeta, y) e^{-y} \right) \Big] \\ & + 2\eta \int_0^\infty d\zeta \left[\zeta^2 \int_\zeta^\infty dy \frac{\hat{r}_\parallel^D(\zeta, y)}{e^y - \hat{r}_\parallel^D(\zeta, y)} + \int_\zeta^\infty y^2 dy \frac{\hat{r}_\perp^D(\zeta, y)}{e^y - \hat{r}_\perp^D(\zeta, y)} \right] \\ & \left. - 2\eta^2 \int_0^\infty d\zeta \left[\zeta^4 \int_\zeta^\infty dy \frac{e^y \hat{r}_\parallel^D(\zeta, y)}{y(e^y - \hat{r}_\parallel^D(\zeta, y))^2} + \int_\zeta^\infty y^3 dy \frac{e^y \hat{r}_\perp^D(\zeta, y)}{(e^y - \hat{r}_\perp^D(\zeta, y))^2} \right] \right\}. \end{aligned} \quad (63)$$

Here, the reflection coefficients for dielectric with the frequency-dependent dielectric permittivity are defined in Eq. (56). For many dielectrics, admitting the presentation (50) with one oscillator, the characteristic frequency at typical separations is much less than the absorption frequency leading to $b = \omega_c/\omega_1 \ll 1$. In fact the small parameter b is of order of another small parameter η . The expansion of Eq. (56) in powers of b takes the form

$$\begin{aligned} \hat{r}_\parallel^D(x, y) &= r_\parallel^D(x, y) - b^2 \frac{(\varepsilon_0^D - 1)x^2 y [2y^2 + (\varepsilon_0^D - 2)x^2]}{\sqrt{(\varepsilon_0^D - 1)x^2 + y^2} \left(\varepsilon_0^D y + \sqrt{(\varepsilon_0^D - 1)x^2 + y^2} \right)^2} \\ &\quad + O(b^4), \\ \hat{r}_\perp^D(x, y) &= r_\perp^D(x, y) - b^2 \frac{(\varepsilon_0^D - 1)x^4 y}{\sqrt{(\varepsilon_0^D - 1)x^2 + y^2} \left(y + \sqrt{(\varepsilon_0^D - 1)x^2 + y^2} \right)^2} \\ &\quad + O(b^4), \end{aligned} \quad (64)$$

where $r_{\parallel,\perp}^D(x, y)$ are the reflection coefficients for dielectric with a frequency-independent dielectric permittivity ε_0^D . Our goal is to obtain the expansion of $\hat{E}(a)$ up to the second powers in the small parameters η and b .

To attain this goal, we note that Eq. (64) contains the zeroth and second powers in b . Thus, both of them should be substituted in the zeroth power

in η in Eq. (63). Considering the terms of order η and η^2 in Eq. (63) we should restrict ourselves by only zeroth order in b , i.e., replace $\hat{r}_{\parallel,\perp}(x, y)$ for $r_{\parallel,\perp}(x, y)$. The calculational scheme of all coefficients accompanying η , η^2 and b^2 is the same as was used in Section 3 for obtaining the analytic expression for the function $\psi_{DM}(\varepsilon_0^D)$. It consists in the expansion of the integrands in a power series, changing the order of integrals and introducing the new variable $w = \zeta/y$. In the order η^0 in Eq. (63) we obtain the contribution already calculated in Eqs. (39), (44) and the contribution of order b^2 . The latter takes the form

$$\begin{aligned} \hat{E}_{b^2}(a) = & \frac{\hbar c b^2 (\varepsilon_0^D - 1)}{32\pi^2 a^3} \sum_{n=1}^{\infty} \int_0^{\infty} dy y^4 e^{-ny} \int_0^1 dw \frac{w^2}{\sqrt{(\varepsilon_0^D - 1)w^2 + 1}} \\ & \times \left\{ \frac{[2 + (\varepsilon_0^D - 2)w^2] [r_{\parallel}^D(w)]^{n-1}}{[\varepsilon_0^D + \sqrt{(\varepsilon_0^D - 1)w^2 + 1}]^2} + \frac{w^2 [r_{\perp}^D(w)]^{n-1}}{[1 + \sqrt{(\varepsilon_0^D - 1)w^2 + 1}]^2} \right\}. \end{aligned} \quad (65)$$

After the integration with respect to y and summation we arrive at

$$\begin{aligned} \hat{E}_{b^2}(a) = & \frac{3\hbar c b^2}{4\pi^2 a^3} \int_0^1 dw \frac{w^2}{\sqrt{(\varepsilon_0^D - 1)w^2 + 1}} \\ & \times \left\{ \frac{2 + (\varepsilon_0^D - 2)w^2}{\varepsilon_0^D + 1 - w^2} \text{Li}_5[r_{\parallel}^D(w)] + \text{Li}_5[r_{\perp}^D(w)] \right\}. \end{aligned} \quad (66)$$

Following the same procedure for the terms of order η in Eq. (63) we obtain

$$\hat{E}_{\eta}(a) = \frac{3\hbar c \eta}{8\pi^2 a^3} \int_0^1 dw \left\{ w^2 \text{Li}_4[r_{\parallel}^D(w)] + \text{Li}_4[r_{\perp}^D(w)] \right\}. \quad (67)$$

Quite analogically for the terms of order η^2 in Eq. (63) it follows

$$\hat{E}_{\eta^2}(a) = -\frac{3\hbar c \eta^2}{2\pi^2 a^3} \int_0^1 dw \left\{ w^4 \text{Li}_4[r_{\parallel}^D(w)] + \text{Li}_4[r_{\perp}^D(w)] \right\}. \quad (68)$$

By combining Eqs. (39) and (66)–(68) one arrives at the Casimir energy in the configuration of metal-dielectric plates made of real materials,

$$\hat{E}(a) = -\frac{\pi^2 \hbar c}{720 a^3} \psi_{DM}(\varepsilon_0^D) \left[1 - C_1(\varepsilon_0^D) \frac{\delta}{a} + C_2(\varepsilon_0^D) \frac{\delta^2}{a^2} - B(\varepsilon_0^D) \frac{\omega_c^2}{\omega_1^2} \right], \quad (69)$$

where the positive coefficients C_1 , C_2 and B are defined as

$$\begin{aligned} C_1(\varepsilon_0^D) \frac{\delta}{a} &\equiv -\frac{\hat{E}_\eta(a)}{E(a)}, & C_2(\varepsilon_0^D) \frac{\delta^2}{a^2} &\equiv \frac{\hat{E}_{\eta^2}(a)}{E(a)}, \\ B(\varepsilon_0^D) \frac{\omega_c^2}{\omega_1^2} &\equiv -\frac{\hat{E}_{b^2}(a)}{E(a)}, \end{aligned} \quad (70)$$

and $E(a)$ is given in Eq. (39).

In Fig. 1 the above coefficients are plotted as functions of ε_0^D by the long-dashed lines 1 and 2 (C_1 and C_2 , respectively) and by the short-dashed line (B). From Eq. (69) and Fig. 1 it is easy to obtain the respective analytic expression for the Casimir pressure,

$$\hat{P}_0(a) = -\frac{\pi^2 \hbar c}{240 a^4} \psi_{DM}(\varepsilon_0^D) \left[1 - \frac{4}{3} C_1(\varepsilon_0^D) \frac{\delta}{a} + \frac{5}{3} C_2(\varepsilon_0^D) \frac{\delta^2}{a^2} - \frac{5}{3} B(\varepsilon_0^D) \frac{\omega_c^2}{\omega_1^2} \right]. \quad (71)$$

Equations (69) and (71) give the possibility to simply find the Casimir energy and pressure between metal and dielectric with rather high precision (see the next section).

From Eqs. (12) and (62) we obtain the asymptotic behavior of the Casimir entropy at small τ for metal-dielectric plates made of real materials,

$$\begin{aligned} \hat{S}(a, T) &= \frac{3k_B \zeta(3)(\varepsilon_0^D - 1)^2}{128\pi^3 a^2 (\varepsilon_0^D + 1)} \tau^2 \left\{ 1 - \frac{\pi^2 (\varepsilon_0^D + 1)}{45 \zeta(3) (\varepsilon_0^D - 1)} \tau \right. \\ &\quad \times \left[\frac{8 \left(1 - 2\varepsilon_0^D \sqrt{\varepsilon_0^D} + (\varepsilon_0^D)^2 \sqrt{\varepsilon_0^D} \right)}{3(\varepsilon_0^D - 1)} - (5\varepsilon_0^D + 11)\eta \right] + O(\tau^2) \left. \right\}. \end{aligned} \quad (72)$$

As is seen from Eq. (72), $\hat{S}(a, T)$ goes to zero when the temperature vanishes as is required by the Nernst heat theorem. This completes the proof of the important statement that the Lifshitz theory in the configuration metal-dielectric is consistent with thermodynamics if the static dielectric permittivity of a dielectric plate is finite.

We complete this section by the consideration of the high-temperature limit. Here the zero-frequency term (46) of the Lifshitz formula determines the total result. In the configuration of metal-dielectric plates only the transverse magnetic mode (for which the metal reflection coefficient is equal to unity) contributes to the zero-frequency term. As a result, unlike the case of two

metal plates, finite conductivity corrections do not contribute at large separations (high temperatures). Thus, for metal and dielectric plates made of real materials, equations (47) and (48) obtained for ideal metal and dielectric with constant permittivity preserve their validity.

5. Comparison between analytic and numerical results

Here we compare the analytic results for the Casimir energy, free energy and pressure given by Eqs. (69), (71), (62) and (37) with the results of numerical computations using the Lifshitz formulas (14) and the dielectric permittivities $\varepsilon^{M,D}(i\xi_l)$ determined from the tabulated optical data for the complex index of refraction. This comparison permits us to find the applicability regions of the obtained analytic results for different materials. As an example we consider the metal plate made of Au and the dielectric plate made of high-resistivity Si.

The most precise results for $\varepsilon^M(i\xi_l)$ in the case of Au were obtained in [23] and for $\varepsilon^D(i\xi_l)$ in the case of Si in [62]. In both cases the data for $\text{Im } \varepsilon^{M,D}(\omega)$ were taken from [63] and the dielectric permittivities along the imaginary frequency axis were computed by means of the Kramers-Kronig relation,

$$\varepsilon^{M,D}(i\xi_l) = 1 + \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega \text{Im} \varepsilon^{M,D}(\omega)}{\omega^2 + \xi^2}. \quad (73)$$

Note that the dielectric permittivity of Si along the imaginary frequency axis is equal to its static value ($\varepsilon_0^D = 11.66$) up to the angular frequency of 5×10^{14} rad/s and with increase of frequency decreases to unity. The analytical results were computed with the plasma frequency of Au equal to $\omega_p = 9.0$ eV and the characteristic absorption frequency of Si equal to $\omega_1 = 4.2$ eV [63] ($1 \text{ eV} = 1.519 \times 10^{15}$ rad/s).

In Fig. 2 we compare the results of analytic and numerical computations of the Casimir energy density (A) and pressure (B) at different separations at zero temperature. In the vertical axes the quantities $\delta E = (\hat{E}_a - \hat{E}_n)/\hat{E}_n$ (A) and $\delta P_0 = (\hat{P}_{0,a} - \hat{P}_{0,n})/\hat{P}_{0,n}$ (B) in percent are plotted where \hat{E}_a and $\hat{P}_{0,a}$ are the analytic results calculated by Eqs. (69) and (71), respectively, and \hat{E}_n , $\hat{P}_{0,n}$ are computed numerically using the Lifshitz formula as described above. As is seen in Fig. 2, the largest deviations between the analytic and numerical results (-4.3% and -7.1% for the energy and pressure, respectively) hold at the shortest separation of 100 nm. This is because the plasma model works good only at separations larger than the plasma wavelength. At shorter separations not some analytic representations for ε but the tabulated optical data should

be used to obtain precise results. At separations larger than 200 and 300 nm $|\delta E|$ is less than 0.9% and 0.25%, respectively. As to $|\delta P|$, it is less than 0.9% and 0.25% at respective separations larger than 250 and 370 nm. Thus, the obtained analytic formulas for the Casimir energy density and pressure at zero temperature in between metal and dielectric give rather precise results in a wide separation range with a precision at the fraction of a percent. In some cases this makes unnecessary much more cumbersome numerical computations using the Lifshitz formula and tabulated optical data for the complex index of refraction (note that the use of different sets of tabulated optical data also leads to about 0.5% differences in the numerically computed Casimir forces [17]).

In Fig. 3 the results of analytical and numerical computations of the relative thermal correction to the Casimir energy (A) and pressure (B) at $a = 300$ nm are compared at different temperatures. The relative thermal corrections are defined as

$$\frac{\Delta \hat{\mathcal{F}}(a, T)}{\hat{E}(a)} = \frac{\hat{\mathcal{F}}(a, T) - \hat{E}(a)}{\hat{E}(a)}, \quad \frac{\Delta \hat{P}(a, T)}{\hat{P}_0(a)} = \frac{\hat{P}(a, T) - \hat{P}_0(a)}{\hat{P}_0(a)}, \quad (74)$$

where $\hat{E}(a)$ and $\hat{P}_0(a)$ are the energy density and pressure calculated numerically by using the Lifshitz formulas at zero temperature. The analytical computations of the thermal corrections are performed using Eqs. (62) and (37). Their results are shown by the dashed lines. The numerical computations of the thermal corrections are done with the help of the Lifshitz formula at zero and nonzero temperatures (solid lines). As is seen in Fig. 3A, the low-temperature analytic result for the thermal correction to the energy density reproduces the result of numerical computations at $T \leq 20$ K. From Fig. 3B it follows that the low-temperature analytic expression for the thermal correction to the Casimir pressure works good in a wider temperature region $T \leq 40$ K. The deviations between analytical and numerical results at higher temperatures are explained by the fact that in Eqs. (62) and (37) we have restricted ourselves by only two and one perturbative orders in small parameter τ , respectively. This restriction, however, makes it possible to solve the main problem of our interest which has no numerical solution, i.e., to find the behavior of the Casimir free energy, entropy and pressure at arbitrarily low temperatures.

6. Is the Lifshitz formula for configuration of metal and dielectric consistent with thermodynamics?

In Sections 3–5 it was supposed that at zero frequency the dielectric permittivity of the dielectric plate is finite. It is well known, however, that at nonzero temperature dielectrics possess some dc conductivity $\sigma_0 = \sigma_0(T)$ which is very small in comparison with the conductivity of metals. Usually (see, e.g., [48,64]) this conductivity is included into the model of dielectric response by adding a Drude-like term in the dielectric permittivity of dielectric,

$$\tilde{\varepsilon}^D(i\xi_l) = \varepsilon^D(i\xi_l) + \frac{4\pi\sigma_0(T)}{\xi_l}. \quad (75)$$

Eq. (75) presents the typical example of the situation discussed in Section 4 when the dielectric permittivity depends on the temperature as a parameter. It can be identically represented in the form

$$\tilde{\varepsilon}^D(i\xi_l) = \varepsilon^D(i\xi_l) + \frac{\beta(T)}{l}, \quad (76)$$

where $\beta(T) = 2\hbar\sigma_0(T)/(k_B T)$. The conductivity of dielectrics quickly decreases with temperature, $\sigma_0(T) \sim \exp(-g/T)$, where the coefficient g is determined by the width of the energy gap Δ [65]. The magnitude of the additional term $\beta(T)/l$ in Eq. (76) is very small. Thus, for SiO_2 at $T = 300$ K it holds $\beta \sim 10^{-12}$ [66]. This makes the role of dielectric dc conductivity negligible at all $l \geq 1$.

The question arises on the possible role of dielectric dc conductivity in the Casimir interaction between metal and dielectric. As was shown in [45,46,47], the Lifshitz formula for the Casimir interaction between two dielectrics cannot incorporate the effects of the dc conductivity because then an inconsistency with thermodynamics arises. The substitution of Eq. (75) in Eq. (10) leads to

$$\tilde{r}_{\parallel}^D(0, y) = 1, \quad \tilde{r}_{\perp}^D(0, y) = 0, \quad (77)$$

instead of Eq. (15). Despite the negligible role of the dc conductivity at all $l \geq 1$, this could lead to important consequences for the Casimir interaction between metal and dielectric. Importantly, the inclusion of the dc conductivity of a dielectric plate leads to a discontinuity in the transverse magnetic reflection coefficient at zero frequency as is seen from Eqs. (15) and (77). This is unlike the case with metals described by the Drude model where the discontinuity arises in the transverse electric reflection coefficient at zero frequency.

To investigate this problem, we substitute the dielectric permittivity (76) in Eq. (9) instead of $\varepsilon^D(i\xi_l)$ defined in Eq. (50). For a metal, as in Section 4, the dielectric permittivity in Eq. (49) is used. In such a way the Casimir free energy $\tilde{\mathcal{F}}(a, T)$ is obtained which takes into account the effects of the dielectric dc conductivity. It is convenient to separate the zero-frequency term of $\tilde{\mathcal{F}}(a, T)$ and subtract and add the zero-frequency term of the free energy $\mathcal{F}(a, T)$ calculated with the dielectric permittivity $\varepsilon^D(i\xi_l)$:

$$\begin{aligned}\tilde{\mathcal{F}}(a, T) = & \frac{k_B T}{16\pi a^2} \int_0^\infty y dy \left[\ln(1 - e^{-y}) - \ln(1 - r_0 e^{-y}) \right] \\ & + \frac{k_B T}{16\pi a^2} \int_0^\infty y dy \ln(1 - r_0 e^{-y}) \\ & + \frac{k_B T}{8\pi a^2} \sum_{l=1}^\infty \int_{\zeta_l}^\infty y dy \left\{ \ln \left[1 - \hat{r}_\parallel^M(\zeta_l, y) \tilde{r}_\parallel^D(\zeta_l, y) e^{-y} \right] \right. \\ & \left. + \ln \left[1 - \hat{r}_\perp^M(\zeta_l, y) \tilde{r}_\perp^D(\zeta_l, y) e^{-y} \right] \right\}.\end{aligned}\tag{78}$$

Here the reflection coefficients $\tilde{r}_{\parallel, \perp}^D$ are found by using Eq. (10) where the dielectric permittivities $\varepsilon^D(i\xi_l)$ from Eq. (50) are replaced by $\tilde{\varepsilon}^D(i\xi_l)$ from Eq. (76).

To find the behavior of $\tilde{\mathcal{F}}(a, T)$ at low temperatures, we expand the last integral on the right-hand side of Eq. (78) in powers of the small parameter $\beta(T)/l$. The zero-order contribution in this expansion together with the second integral on the right-hand side of Eq. (78) are equal to the Casimir free energy $\mathcal{F}(a, T)$ calculated with dielectric permittivity $\varepsilon^D(i\xi_l)$. Calculating explicitly the first integral on the right-hand side of Eq. (78), we rearrange this equation to the form

$$\tilde{\mathcal{F}}(a, T) = \hat{\mathcal{F}}(a, T) - \frac{k_B T}{16\pi a^2} [\zeta(3) - \text{Li}_3(r_0)] + Q(a, T),\tag{79}$$

where $Q(a, T)$ contains all powers in the expansion of the last integral on the right-hand side of Eq. (78) in the small parameter $\beta(T)/l$ equal or higher than the first one. The explicit expression for the main, linear in $\beta(T)/l$, term in $Q(a, T)$ reads:

$$Q_1(a, T) = \frac{k_B T}{8\pi a^2} \sum_{l=1}^\infty \frac{\beta(T)}{l} \int_{\zeta_l}^\infty \frac{dy y^2 e^{-y}}{\sqrt{y^2 + \zeta_l^2 (\varepsilon_l^D - 1)}}\tag{80}$$

$$\times \left\{ \frac{(2 - \varepsilon_l^D)\zeta_l^2 - 2y^2}{\left[\sqrt{y^2 + \zeta_l^2(\varepsilon_l^D - 1)} + \varepsilon_l^D y\right]^2} \frac{\hat{r}_{\parallel}^M(\zeta_l, y)}{1 - \hat{r}_{\parallel}^M(\zeta_l, y)\hat{r}_{\parallel}^D(\zeta_l, y)e^{-y}} - \frac{\zeta_l^2}{\left[\sqrt{y^2 + \zeta_l^2(\varepsilon_l^D - 1)} + y\right]^2} \frac{\hat{r}_{\perp}^M(\zeta_l, y)}{1 - \hat{r}_{\perp}^M(\zeta_l, y)\hat{r}_{\perp}^D(\zeta_l, y)e^{-y}} \right\}.$$

To determine the asymptotic behavior of Eq. (80) when $\tau \rightarrow 0$, we expand the integrated function in powers of τ (recall that $\zeta_l = \tau l$) and consider the main contribution in this expansion at $\tau = 0$:

$$\begin{aligned} Q_1(a, T) &= -\frac{k_B T r_0}{4\pi a^2 \left((\varepsilon_0^D)^2 - 1\right)} \sum_{l=1}^{\infty} \frac{\beta(T)}{l} \int_{\zeta_l}^{\infty} \frac{y dy e^{-y}}{1 - r_0 e^{-y}} \\ &= -\frac{k_B T \beta(T)}{4\pi a^2 \left((\varepsilon_0^D)^2 - 1\right)} \sum_{n=1}^{\infty} \frac{r_0^n}{n^2} \left[\sum_{l=1}^{\infty} \frac{e^{-n\tau l}}{l} + n\tau \sum_{l=1}^{\infty} e^{-n\tau l} \right]. \end{aligned} \quad (81)$$

Performing the summation in l we obtain

$$Q_1(a, T) = -\frac{k_B T \beta(T)}{4\pi a^2 \left((\varepsilon_0^D)^2 - 1\right)} \sum_{n=1}^{\infty} \frac{r_0^n}{n^2} \left[-\ln(1 - e^{-n\tau}) + \frac{n\tau}{e^{n\tau} - 1} \right]. \quad (82)$$

The right-hand side of Eq. (82) can be rearranged with the help of the equality

$$-\ln(1 - e^{-n\tau}) + \frac{n\tau}{e^{n\tau} - 1} = -\ln \tau + 1 - \ln n + O(\tau^2). \quad (83)$$

As a result it holds

$$Q_1(a, T) = -\frac{k_B \text{Li}_2(r_0)}{2\pi a^2 \left((\varepsilon_0^D)^2 - 1\right)} T \beta(T) \ln \tau + T \beta(T) O(\tau^0). \quad (84)$$

Taking into account that $\beta(T) \sim (1/T) \exp(-g/T)$, we arrive at

$$Q_1(a, T) \sim e^{-g/T} \ln T. \quad (85)$$

From Eq. (85) it follows that both $Q_1(a, T)$ and its derivative with respect to T go to zero when T vanishes. The terms of higher orders in the small parameters $\beta(T)/l$ and τ omitted in our analysis, go to zero even faster than

Q_1 . Thus, the quantity $Q(a, T)$ in Eq. (79) and its derivative with respect to T have zero limits when the temperature goes to zero.

Now we are in a position to find the asymptotic behavior of the entropy in the configuration metal-dielectric with included dc conductivity of the dielectric plate. Using Eq. (12), we obtain from Eq. (79)

$$\tilde{S}(a, T) = \hat{S}(a, T) + \frac{k_B}{16\pi a^2} [\zeta(3) - \text{Li}_3(r_0)] - \frac{\partial Q(a, T)}{\partial T}, \quad (86)$$

where $\hat{S}(a, T)$ is defined in Eq. (72). In the limit $T \rightarrow 0$ Eq. (86) results in

$$\tilde{S}(a, T) = \frac{k_B}{16\pi a^2} [\zeta(3) - \text{Li}_3(r_0)] > 0. \quad (87)$$

This equation implies that in the configuration of metal-dielectric with included dc conductivity of the dielectric plate the Nernst heat theorem is violated. Previously the analogous result was obtained [45,46,47] for the configuration of two dielectric plates with frequency-dependent dielectric permittivities. It is easily seen, that Eq. (87) is preserved, if, instead of the plasma dielectric model (49), the Drude dielectric function is used in the case of metal plate with a perfect crystal lattice. If the metal plate has impurities, the analytical derivation cannot be performed, but numerical computations lead to the same positive value of the entropy as in Eq. (87). Thus, the Lifshitz theory becomes inconsistent with thermodynamics when the dc conductivity of a dielectric plate is taken into account. This suggests that the actual low-frequency behavior of the dielectric properties is not related to the phenomena of van der Waals and Casimir forces and should not be included into the model of dielectric response.

Recently [28] this theoretical conclusion was confirmed experimentally in the measurement of the difference Casimir force acting between Au-coated sphere and Si plate illuminated by laser pulses. The difference of the Casimir forces in the presence and in the absence of pulse was measured using an atomic force microscope. In the absence of laser pulse the concentration of charge carriers was of about $5 \times 10^{14} \text{ cm}^{-3}$ (higher-resistivity Si), but in the presence of pulse this concentration has been enhanced up to $2 \times 10^{19} \text{ cm}^{-3}$ (lower-resistivity Si). The experimental data were compared with two theories. The first theory used an assumption that in the absence of laser light Si possesses a finite static dielectric permittivity $\varepsilon_0^D = 11.66$ (see Section 5). The second theory took into account the dc conductivity of Si in the absence of laser light like it was done above in Section 6. The first theory was found to be in excellent agreement with data, whereas the second theory was excluded at the 95% confidence level within the separation region from 100 to 200 nm [28]. Thus, the inclusion of the dc conductivity of dielectrics and high-resistivity semiconductors in the

model of dielectric response is not only inconsistent thermodynamically but is also in contradiction with experiment.

7. Conclusions and discussion

In this paper we have obtained the analytic expressions for the Casimir free energy, pressure and entropy at low temperatures in the configuration of one metal and one dielectric plate. Different models of the dielectric response for both metal and dielectric were considered: the simplified model of an ideal metal and dielectric with constant dielectric permittivity, and the realistic model of a metal described by the plasma model and a dielectric described in the Ninham-Parsegian representation for ε . To derive the asymptotic expressions at low temperatures, the perturbation theory in the small parameter τ was developed which is proportional to the product of separation distance and the temperature. The analytic expressions for the main physical quantities in the limit of high temperatures and at zero temperature were also obtained. The analytic results were compared with numerical computations using the Lifshitz formula and tabulated optical data, and good agreement was found.

The fundamental conclusion arrived in the paper is that the Lifshitz theory applied to the configuration of one metal and one dielectric plate is in agreement with thermodynamics if the dielectric permittivity of a dielectric plate at zero frequency is finite. In particular, it was shown that the Casimir entropy goes to zero when the temperature vanishes, i.e., the Nernst heat theorem is satisfied. This conclusion cannot be reached numerically, because it is impossible to perform numerical computations at arbitrarily low temperature and their precision is always restricted. On the contrary, it was shown that, if the dielectric permittivity of the dielectric plate at zero frequency turns to infinity (i.e., small dc conductivity of a dielectric material is taken into account), this leads to a nonzero value of the Casimir entropy at zero temperature, i.e., to a violation of the Nernst heat theorem. The inclusion of the dc conductivity of high-resistivity semiconductors in the model of dielectric response was also recently shown to be in contradiction with experiment [28]. What this means is that to avoid contradictions with thermodynamics and experiment, one should not include the actual conductivity properties of dielectric materials at very low, quasistatic, frequencies into the model of the dielectric response (this phenomenological prescription was obtained previously in [45,46,47] for the case of two dielectric plates made of real materials).

The above conclusions can be discussed in the context of the formalism of thermal quantum field theory in Matsubara formulation where the zero-frequency term plays a separated role and calls for an adequate interpretation. It is common knowledge that the zero-point energy of quantized fields contains os-

cillations of any frequency. It would be hard, however, to imagine the presence of a zero-frequency (i.e., constant) field in the vacuum state. This suggests that the zero-frequency term in the Matsubara summation could be understood not literally but as a mathematical limit to zero from the region of much higher frequencies [the characteristic frequency of the Casimir effect $c/(2a)$, and the thermal frequency $k_B T/\hbar$] which determine the physical phenomena of the van der Waals and Casimir forces. As was discussed below Eq. (76), in the region of characteristic and thermal frequencies the effects of dc conductivity contribute twelve orders of magnitude less than ε^D . Because of this, the dc conductivity may be considered as not related to the Casimir forces and be not included into the zero-frequency term of the Lifshitz formula. This phenomenological prescription is not the fundamental resolution of the problem, which remains unknown, but following it one avoids contradictions to thermodynamics and experiment.

Note that the problems discussed above for the configuration of metal-dielectric are of different nature than those arising for two metals. For metals the concentration of charge carriers only slightly depends on the temperature. The validity of the Nernst heat theorem in the Casimir interaction between two metals is caused by the scattering processes of free charge carriers on phonons, impurities etc. For the perfect Drude metals with no impurities, relaxation goes to zero when the temperature vanishes and the Nernst heat theorem is violated [33,43,44]. On the contrary, for dielectrics the concentration of charge carriers quickly decreases to zero when the temperature vanishes. Here the violation of the Nernst heat theorem does not depend on the scattering processes and is caused by the inclusion of the infinitely large dc conductivity. In the formalism, for two metals a discontinuity in the reflection coefficient of the transverse electric mode at zero frequency arises, whereas for a metal and dielectric a discontinuity holds in the reflection coefficient of the transverse magnetic mode. These differences are reflected in the fact that even the sign of the entropy at $T = 0$ for two perfect Drude metals and for one metal and one dielectric with included dc conductivity are opposite (negative for two metals and positive for a metal and dielectric).

The obtained results are topical for the interpretation of recent measurements of the Casimir force between metal sphere and semiconductor plate [25,26,27,28]. Semiconductors suggest a wide variety of the conductivity properties ranging from metallic to dielectric ones. In the application of the Lifshitz theory to the metal-semiconductor test bodies the model of the dielectric response for a semiconductor should be chosen to satisfy the Nernst heat theorem and other fundamental physical principles. This can be done by using the proposed phenomenological prescription. A more fundamental resolution of the discussed problems may go beyond the scope of the Lifshitz theory.

Acknowledgments

G. L. K. and V. M. M. are grateful to the Center of Theoretical Studies and the Institute for Theoretical Physics, Leipzig University for their kind hospitality. This work was supported by Deutsche Forschungsgemeinschaft, Grant No. 436 RUS 113/789/0-2.

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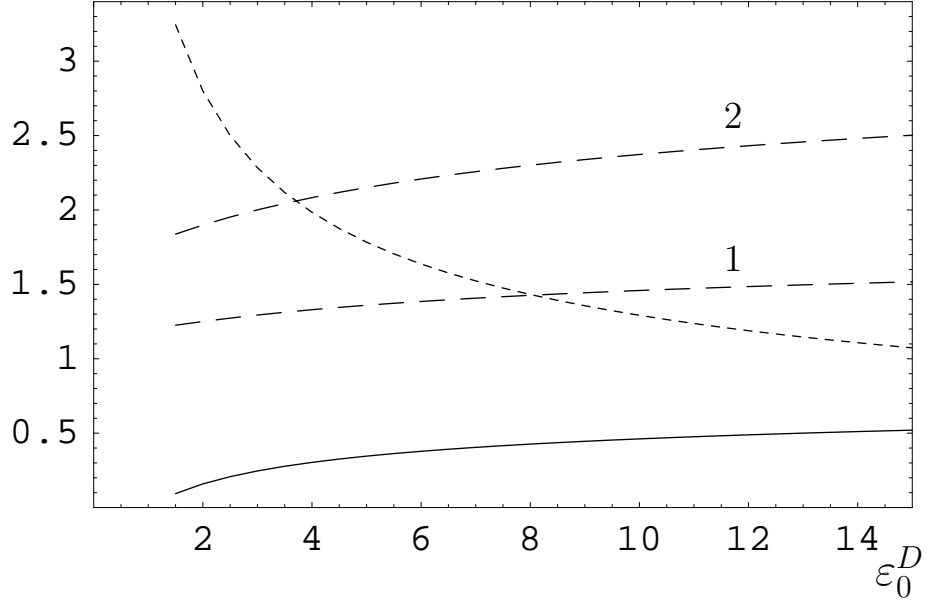


Fig. 1. The correction factor ψ_{DM} to the Casimir energy density (solid line) as a function of the static dielectric permittivity. The long-dashed lines 1, 2 and the short-dashed line show the coefficients C_1 , C_2 and B , respectively, in Eq. (69) for the Casimir energy density between plates made of real metal and dielectric.

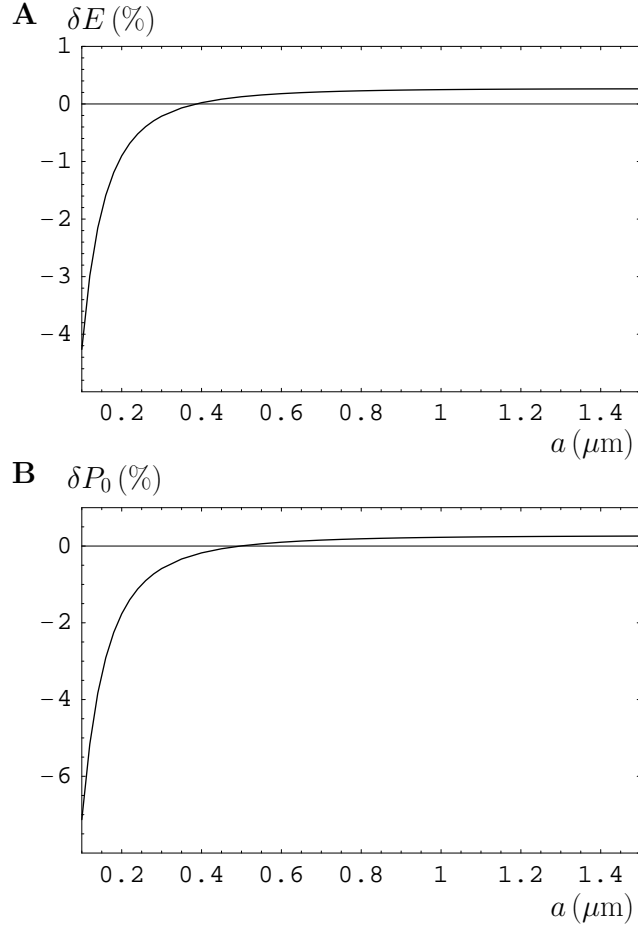


Fig. 2. The relative differences between the analytic and numerical results for the Casimir energy density (A) and pressure (B) at zero temperature versus separation.

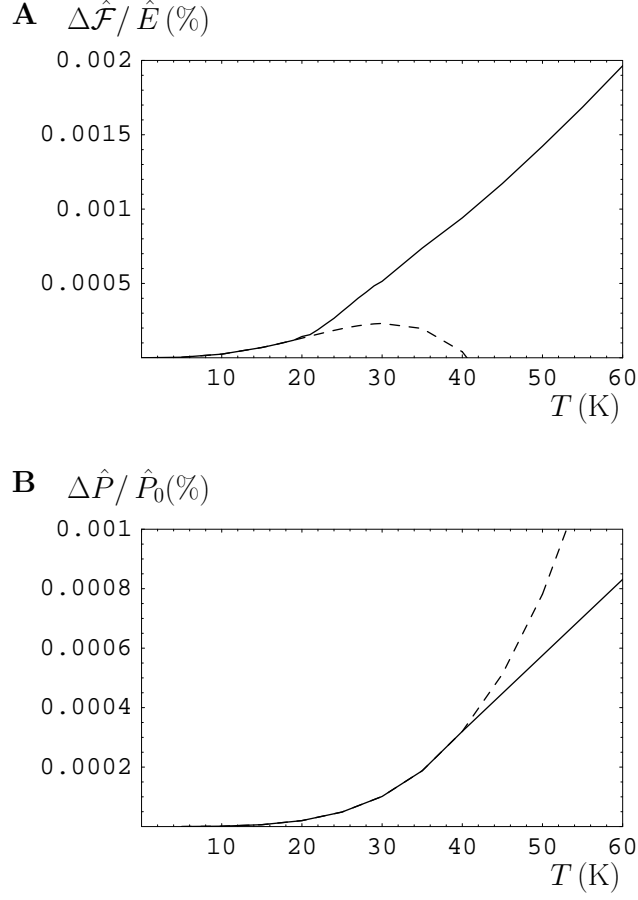


Fig. 3. The relative thermal correction to the Casimir energy density (A) and pressure (B) versus temperature at a separation $a = 300$ nm. Solid lines show the results of numerical computations and the dashed lines are obtained using the analytic asymptotic expressions.